Patterns in a Smoluchowski Equation

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Abstract

We analyze the dynamics of concentrated polymer solutions modeled by a 2D Smoluchowski equation. We describe the long time behavior of the polymer suspensions in a fluid.

When the flow influence is neglected the equation has a gradient structure. The presence of a simple flow introduces significant structural changes in the dynamics. We study the case of an externally imposed flow with homogeneous gradient. We show that the equation is still dissipative but new phenomena appear. The dynamics depend on both the concentration intensity and the structure of the flow. In certain *limit cases* the equation has a gradient structure, in an appropriate reference frame, and the solutions evolve to either a steady state or a tumbling wave. For small perturbations of the gradient structure we show that some features of the gradient dynamics survive: for small concentrations the solutions evolve in the long time limit to a steady state and for high concentrations there is a tumbling wave.

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1 Introduction

In this paper we study qualitative properties of a Smoluchowski equation describing the dynamics of non-Newtonian complex fluids containing liquid crystalline polymers, in a concentrated regime.

The model we use was introduced by M. Doi in ([7]) (see also [8]). It identifies the polymers with inflexible rods whose thickness is much smaller than their length. We study here the case in which the fluid is two dimensional. This represents a simplification that preserves many of the qualitative features of the physical three dimensional phenomenon ([22]). This model as well as its three dimensional analogue has attracted much interest in the recent years ([2],[3], [5], [6],[32],[12],[13],[14] ,[19],[29],[31],[34]).

The local probability measure associated with the polymers is of the form $f(t, x, \theta)d\theta$. Here t is the time coordinate, $x \in \mathbb{R}^2$ denotes the spatial coordinate and $\theta \in [0, 2\pi]$ is a direction on the unit circle. The measure $f d\theta$ represents the time-dependent probability

that a rod with center of mass at x has an axis of direction θ in the area element $d\theta$. The equation we study, in non-dimensional form, is:

$$\partial_t f + u \cdot \nabla_x f + \partial_\theta (Vf + \partial_\theta(\mathcal{K}f) \cdot f) = \partial_{\theta\theta} f \tag{1}$$

where u is the velocity of the underlying fluid. For $\nabla_x u = (u_j^i)_{i,j=1,2}$,

$$V(x,\theta) = -u_1^1(x)\cos\theta\sin\theta - u_2^1(x)\sin^2\theta + u_1^2(x)\cos^2\theta + u_2^2(x)\sin\theta\cos\theta \tag{2}$$

denotes the projection of $\nabla_x u \cdot (\cos(\theta), \sin(\theta))^t$ on the tangent space in $(\cos(\theta), \sin(\theta))^t$. This term describes the way the fluid influences the evolution of f.

The term $\mathcal{K}f$ – the excluded volume potential, which accounts for the interaction between different rods – is given by

$$\mathcal{K}f(\theta) = b \int_{S^1} k(\theta - \theta')f(\theta')d\theta' \tag{3}$$

where k is a smooth function and b is a non-dimensional parameter measuring the concentration of the polymers in the fluid. Moreover

$$k(\theta) = k(-\theta) \tag{4}$$

In many instances we will restrict ourselves to using $k(\theta) = \frac{1}{2}\cos(2\theta)$ in which case $\mathcal{K}f$ is the so called Maier-Saupe potential. This potential has been frequently used in the literature ([3], [5], [6]).

The fluid velocity u obeys the Stokes or the Navier-Stokes equations forced by an appropriate average of f ([8]).

The rich dynamical behavior of the system poses significant numerical and analytical challenges ([25], [26]). We consider two levels of complexity:

On a first level one neglects the influence of the fluid. This situation has been analyzed in ([3], [5]). It was shown that the system has a gradient structure and it is dissipative, i.e. the solutions starting from an arbitrary initial data end up in a fixed ball. We refine this analysis by showing that the ω-limit set of any solution consists of steady states. We also show that ω-limit set is reduced to only one steady state if some additional symmetry constraints are imposed.

In the presence of the flow the dynamics become very complex, even when the flow has a simple structure. We consider the case of an externally imposed flow with homogeneous gradient, i.e. the matrix $\nabla_x u$ is a given constant matrix. This is a situation of physical importance because it captures the local behavior of steady, smooth flows, and hence it is encountered frequently in the rheological literature (see [1], chapter 5). A similar situation, when the fluid is a shear flow and has small amplitude, has been recently analyzed in [33].

We prove that the equation is dissipative in this case as well. It turns out that the dynamics depend strongly on both the intensity of the concentration b and the flow structure. Let us observe that in the case of homogeneous gradient flow V is independent of x and we can write

$$V = \omega + s\cos(2\theta + \alpha)$$

with $\omega \in \mathbb{R}, s > 0$ uniquely determined and α determined modulo 2π. One can easily check that $\omega = \frac{u_1^2 - u_2^1}{2}$ so ω is the vorticity of the imposed flow. We can assume without

loss of generality that $\alpha = 0$ (if $\alpha \neq 0$ then $\tilde{f}(t, \theta) = f(t, \theta - \alpha/2)$ satisfies the same equation but with $\cos(2\theta + \alpha)$ in V replaced by $\cos(2\theta)$). The parameters ω and s determine the flow structure.

In certain *limit cases* the equation preserves a gradient structure in the initial reference frame (if $\omega = 0$) or in a rotating frame (if $s = 0$). The parameters ω and s have very different rôles. One can think, heuristically, that s determines the long time shape of the solution while ω affects the rotational behavior of the solution.

In general though, when both ω and s are non-zero there is no obvious gradient structure but some of the gradient dynamics features are still present. For flows with arbitrary $\omega \neq 0$ and appropriately small s, at high concentration b, we prove the existence of tumbling solutions, that is solutions which are periodic in time in a moving frame. For arbitrary ω, small s and small b, the solutions evolve to a steady state in the long time limit.

Intriguing and challenging questions remain to be addressed in the framework developed by this paper. For instance, our analysis offers information about the cases when s is small and b is either small or large. We can not offer any information about what happens, for small s, at intermediary values of b. This would the range of b that, in the absence of the flow, corresponds to the transition from isotropic to nematics, from one steady state to two steady states. The case of large s seems to be significantly more difficult.

It is of interest to determine how general is the behavior described above. Indeed, for quite general $1D$ *local* nonlinear parabolic equations with bounded trajectories it is known that the solutions evolve in the long time limit to either a steady state or a periodic solution ([10],[27]). The presence of a non-local term may allow for significantly more complicated dynamics ([11]). We do not know if this is the case in here or one can prove a Poincaré-Bendixson theorem as in ([10]).

The paper is organized as follows: in the next section we consider a general nonlinear Fokker-Planck equation and prove its dissipativity. We recognize that both levels of complexity can be put into this general form and this gives us the dissipativity at both levels. Moreover, when the nonlinear Fokker-Planck equation has a gradient structure we analyze its ω-limit sets. We obtain thus information about the case when the flow is neglected or the flow is present but irrotational.

In the last section we consider the case when the flow is present and is externally imposed, of constant gradient. If s and b are small we prove the evolution to a steady state. The proof also shows the existence of a unique steady state by using arguments from the dynamical systems theory.

Finally, we prove our main theorem: the existence of time periodic solutions in the appropriate moving frame for small s and large b. The proof involves the use of a non-standard form of the implicit function theorem in Banach spaces.

2 The dissipativity

2.1 The general case

Consider the nonlinear Fokker-Planck equation:

$$\partial_t f + \partial_\theta(fV + f\partial_\theta \mathcal{K}f) = \partial_{\theta\theta} f \tag{5}$$

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where f depends only on time and $\theta \in [0, 2\pi]$ with V a potential which depends on $\theta \in [0, 2\pi]$ and may depend on time i.e. $V = V(t, \theta)$. We assume that $f(t, \cdot)$ is periodic $f(t, 0) = f(t, 2\pi), \forall t \geq 0$ and since f represents a probability distribution function we assume that for all $t \geq 0$ it is positive and of mean one. We also assume that V is smooth, periodic in the θ variable and $\mathcal{K}f$ is defined as in (3).

We prove that any solution of the equation (5) which starts from a nonnegative, mean one, smooth initial data will eventually enter a fixed ball in H^1. Let us remark that for the case when $V = 0$, $\mathcal{K}f$ is the Maier-Saupe potential and even initial data the existence of a global attractor in any Sobolev norm, was proved in [5].

Theorem 1 *(The dissipativity) Assume that there exist M, N such that*

$$\|\partial_\theta V(t, \theta)\|_{L^\infty(S^1)} \leq M, \forall t \geq 0 \ |\partial_{\theta\theta}k| \leq N, \forall \theta$$

Let f be a solution of (5) starting from a smooth nonnegative initial data, with $\|f(0)\|_{L^1} = 1$. We have

$$\|f(t)\|_{L^2}^2 \leq \|f(0)\|_{L^2}^2 e^{-t/2} + \bar{C} + \frac{1}{\pi} \tag{6}$$

$$\|\partial_\theta f(t)\|_{L^2}^2 \leq \left(\|\partial_\theta f(0)\|_{L^2}^2 + \|f(0)\|_{L^2}^2 \cdot 6(M + bN)\right) e^{-t/4} + (M + bN)\left(8\bar{C} + \frac{6}{\pi}\right) \tag{7}$$

with

$$\bar{C} = 4\left(\frac{M + bN}{2}(\frac{1}{2\pi} + C2^{1/3}) + (\frac{MC}{2^{2/3}})^2 + (\frac{bNC}{2^{2/3}})^2\right)$$

where C is a constant that appears in the Gagliardo-Nirenberg inequality (8).

Proof. The existence of solutions is obtained by standard arguments (see also [3]). The positivity of the initial data is preserved, by the maximum principle, and since $\frac{d}{dt}\int_{S^1} f(t, \theta)d\theta = 0$ we have that $\|f(t, \cdot)\|_{L^1} = 1, \forall t \geq 0$.

Let us recall the classical Gagliardo-Nirenberg inequalities ([28]) which state that for $1 \leq q, r \leq \infty$ and j, m integers satisfying $0 \leq j < m$ and for f an appropriately smooth, periodic, mean zero function

$$\|D^j f\|_{L^p} \leq C\|D^m f\|_{L^r}^a\|f\|_{L^q}^{1-a} \tag{8}$$

with

$$\frac{1}{p} = \frac{j}{d} + a(\frac{1}{r} - \frac{m}{d}) + \frac{1-a}{q}$$

where d is the spatial dimension.

Taking $d = 1, j = 0, p = \infty, m = 1, r = 2, q = 1$ we obtain

$$\|f - \bar{f}\|_{L^\infty} \leq C\|\partial_\theta f\|_{L^2}^{2/3}\|f - \bar{f}\|_{L^1}^{1/3}$$

where we denoted by $\bar{f}$ the average of f. Using the fact that the L^1 norm of f is 1 and $\bar{f} = \frac{1}{2\pi}$ the last inequality implies

$$\|f\|_{L^\infty} \leq \frac{1}{2\pi} + C2^{1/3}(1 + \|\partial_\theta f\|_{L^2}) \tag{9}$$

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Also, using Poincaré's inequality and $\bar{f} = \frac{1}{2\pi}$ we have

$$\|f\|_{L^2} \le \|\bar{f}\|_{L^2} + \|f - \bar{f}\|_{L^2} \le \frac{1}{\sqrt{2\pi}} + \|\partial_\theta f\|_{L^2}$$

and then

$$\|f\|_{L^2}^2 \le \frac{1}{\pi} + 2\|\partial_\theta f\|_{L^2}^2 \tag{10}$$

Multiplying (5) by f, integrating over S^1 and by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{S^1} f^2 + \frac{1}{2}\int_{S^1} f^2 \partial_\theta V + \frac{1}{2}\int_{S^1} (\partial_{\theta\theta}\mathcal{K}f)f^2 = -\int_{S^1}(\partial_\theta f)^2 \tag{11}$$

Using the hypothesis, the bound (9) and $\|f\|_{L^1} = 1$ we get

$$\frac{1}{2}|\int_{S^1}\partial_\theta V f^2 d\theta| \le \frac{1}{2}\|\partial_\theta V\|_{L^\infty}\|f\|_{L^\infty}\|f\|_{L^1}$$

$$\le \frac{M}{2}\left(\frac{1}{2\pi} + C2^{1/3}(1 + \|\partial_\theta f\|_{L^2})\right) \le \frac{M}{2}(\frac{1}{2\pi} + C2^{1/3}) + (\frac{MC2^{1/3}}{2})^2 + \frac{1}{4}\|\partial_\theta f\|_{L^2}^2$$

Similarly

$$\frac{1}{2}|\int_{S^1}(\partial_{\theta\theta}\mathcal{K}f)f^2 d\theta| \le \frac{bN}{2}(\frac{1}{2\pi} + C2^{1/3}) + (\frac{bNC2^{1/3}}{2})^2 + \frac{1}{4}\|\partial_\theta f\|_{L^2}^2$$

Using the last two bounds in (11) we obtain

$$\frac{1}{2}\frac{d}{dt}\|f\|_{L^2(S^1)}^2 \le \underbrace{\frac{M + bN}{2}(\frac{1}{2\pi} + C2^{1/3}) + (\frac{MC}{2^{2/3}})^2 + (\frac{bNC}{2^{2/3}})^2}_{\bar{C}/4} - \frac{1}{2}\|\partial_\theta f\|_{L^2}^2 \tag{12}$$

so then using (10), multiplying by $2e^{t/2}$, integrating on $[0, t]$ and then multiplying by $e^{-t/2}$ we obtain (6).

On the other hand (12) can be rewritten as

$$\frac{1}{2}\frac{d}{dt}\|f\|_{L^2}^2 + \frac{1}{4}\|\partial_\theta f\|_{L^2}^2 \le \frac{\bar{C}}{4} - \frac{1}{4}\|\partial_\theta f\|_{L^2}^2$$

Using (10) on the right hand side, multiplying by $2e^{t/4}$, integrating on $[0, t]$ and multiplying by $e^{-t/4}$ we get

$$\|f(t)\|_{L^2}^2 + \frac{1}{2}\int_0^t \|\partial_\theta f(s)\|_{L^2}^2 e^{(s-t)/4}ds \le 2\bar{C} + \frac{1}{\pi} + \|f(0)\|_{L^2}^2 e^{-t/4} \tag{13}$$

which gives us an apriori bound on a time integral involving $\|\partial_\theta f\|_{L^2}^2$.

In order to obtain the dissipativity of $\|\partial_\theta f\|_{L^2}$ let us differentiate (5) with respect to θ. Denoting $\partial_\theta f = F$ we obtain an equation for F:

$$\partial_t F + \partial_\theta [FV + f\partial_\theta V + F\partial_\theta(\mathcal{K}f) + f\partial_{\theta\theta}(\mathcal{K}f)] = \partial_{\theta\theta}F \tag{14}$$

Multiplying by F, integrating on S^1 and by parts we have

$$\frac{1}{2}\frac{d}{dt}\int_{S^1}F^2 + \frac{1}{2}\int_{S^1}\partial_\theta V F^2 - \int_{S^1}\partial_\theta V f \partial_\theta F$$

$$+\frac{1}{2}\int_{S^1}\partial_{\theta\theta}(\mathcal{K}f)F^2 - \int_{S^1}f\partial_{\theta\theta}(\mathcal{K}f)\partial_\theta F = -\int_{S^1}(\partial_\theta F)^2$$

Hence

$$\frac{1}{2}\frac{d}{dt}\|F\|_{L^2}^2 = \int_{S^1}(\partial_\theta V + \partial_{\theta\theta}(\mathcal{K}f))f\partial_\theta F - \frac{1}{2}\int_{S^1}(\partial_\theta V + \partial_{\theta\theta}(\mathcal{K}f))F^2 - \int_{S^1}(\partial_\theta F)^2$$

$$\leq \frac{1}{2}(\|\partial_\theta V\|_{L^\infty} + b\|\partial_{\theta\theta}k\|_{L^\infty})\int_{S^1}f^2 + \frac{1}{2}\int_{S^1}(\partial_\theta F)^2$$

$$+\frac{1}{2}(\|\partial_\theta V\|_{L^\infty} + b\|\partial_{\theta\theta}k\|_{L^\infty})\int_{S^1}F^2 - \int_{S^1}(\partial_\theta F)^2$$

$$\leq \frac{(M+bN)}{2}(P + \|F\|_{L^2(S^1)}^2) - \frac{\|\partial_\theta F\|_{L^2(S^1)}^2}{2}$$

where for the last inequality we denoted $P = \|f(0)\|_{L^2}^2 e^{-t/2} + \bar{C} + \frac{1}{\pi}$ and used (6) which we have just proved.

Using the fact that $F = f_\theta$ is mean zero and Poincaré's inequality the last inequality implies

$$\frac{d}{dt}\|F\|_{L^2}^2 + \frac{\|F\|_{L^2}^2}{4} \leq (M+bN)(P + \|F\|_{L^2}^2)$$

Multiplying by $e^{t/4}$, integrating on $[0,t]$ and multiplying by $e^{-t/4}$ we have

$$\|F(t)\|_{L^2}^2 \leq \left(\|F(0)\|_{L^2}^2 + 4(M+bN)\|f(0)\|_{L^2}\right)e^{-t/4}$$

$$+(M+bN)\left(4(\bar{C}+\frac{1}{\pi}) + \int_0^t\|F(s)\|_{L^2}^2 e^{(s-t)/4}ds\right)$$

$$\leq (\|F(0)\|_{L^2}^2 + 6(M+bN)\|f(0)\|_{L^2}^2)e^{-t/4} + (M+bN)(8\bar{C}+\frac{6}{\pi})$$

where for the last inequality we used (13). $\square$

Remark 1 *Repeating the procedures described above for higher order derivatives allows one to obtain, inductively, that the equation is dissipative in any Sobolev norm.*

Remark 2 *One can improve the rate of decay into the absorbing ball at the expense of having an absorbing ball of bigger radius.*

Remark 3 *Note that equation (5) generates a compact nonlinear semigroup. Indeed, consider the mapping $S(t) : H^1 \cap \mathcal{C} \cap S_{L^1}(0,1) \to H^1 \cap \mathcal{C} \cap S_{L^1}(0,1)$ which associates to an element f the solution at time t starting from initial data f (we denoted by $\mathcal{C}$ the cone of nonnegative functions and by $S_{L^1}(0,1)$ the unit sphere in $L^1(S^1)$). Then, as $\|f(t)\|_{L^1} = 1, \forall t \geq 0$ and $\mathcal{K}f$ is the convolution of a smooth kernel k with f, we have that $\mathcal{K}f$ is apriori bounded in any Sobolev norm. Thus the equation can be essentially*

treated as a semilinear parabolic equation and standard arguments give the existence, uniqueness and continuous dependence on the initial data which shows that $S(t)$ is a semigroup. The compactness is a consequence of the usual smoothing effect of parabolic equations. See for details [18], Ch.3.

2.2   The gradient case

In the following we assume that there exists a 2π-periodic function $W = W(\theta)$ such that

$$V = \partial_\theta W \tag{15}$$

We have then that (5) becomes an equation of gradient type with the energy functional

$$\mathcal{E} = \int_{S^1} \log f \cdot f - \frac{1}{2} \int_{S^1} \mathcal{K}f \cdot f - \int_{S^1} W \cdot f$$

(see also [4]) and

$$\frac{d\mathcal{E}}{dt} = - \int_{S^1} |\partial_\theta (\log f - \mathcal{K}f - W)|^2 f \, d\theta \tag{16}$$

We show that the presence of this energy functional is enough for proving that the ω-limit set of any solution if made of steady states. We show that the ω-limit reduces to only one steady state if additional symmetry constraints are imposed.

We first need some properties of the energy functional:

Lemma 1 *Assume that $W \in L^\infty(S^1)$. Then the energy functional $\mathcal{E}$ is bounded from below along the solutions and it is locally Lipschitz as a functional from $L^2 \cap \mathcal{C}$ into $\mathbb{R}$ (where $\mathcal{C}$ denotes the cone of nonnegative functions).*

Proof.   The energy $\mathcal{E}(f) = \int_{S^1} f \log f - \frac{1}{2} \int_{S^1} \mathcal{K}f \cdot f - \int_{S^1} fW$ is made of three parts: the (negative) Boltzmann entropy $\int_{S^1} f \log f$, the nonlinear potential contribution $\frac{1}{2} \int_{S^1} \mathcal{K}f \cdot f$ and the linear potential part $\int_{S^1} Wf$.

We have that the nonlinear potential contribution part is bounded in L^∞ thanks to the fact that $f \geq 0$ and $\int_{S^1} f = 1$. Indeed:

$$|| \int_{S^1} \mathcal{K}f \cdot f ||_{L^\infty} \leq ||\mathcal{K}f||_{L^\infty} ||f||_{L^1} \leq ||k||_{L^\infty} (\int_{S^1} f)^2$$

A similar argument works for the linear potential part, using the hypothesis $W \in L^\infty(S^1)$.

On the other hand, the function $x \log x$ is bounded from below by $-\frac{1}{e}$ which combined with the previous observations gives us the boundedness from below of the energy $E(f)$.

The fact that the entropy is a locally Lipschitz functional in L^2 is a consequence of the inequality (see also [16]):

$$|x \log x - y \log y| \leq C(|x - y|^{\frac{1}{2}} + |x - y|(x^{\frac{1}{2}} + y^{\frac{1}{2}}))$$

Also, the nonlinear potential part of the energy is locally Lipschitz in L^2 norm:

$$\|\int \mathcal{K}f \cdot f - \mathcal{K}g \cdot g\|_{L^2} \leq \|\int \mathcal{K}f(f-g) + \int \mathcal{K}(f-g)g\|_{L^2}$$
$$\leq \|\mathcal{K}f\|_{L^\infty}\|f-g\|_{L^2} + \|\mathcal{K}(f-g)\|_{L^\infty}\|g\|_{L^2}$$
$$\leq \|\mathcal{K}\|_{L^1 \to L^\infty}\underbrace{\|f\|_{L^1}}_{=1}\|f-g\|_{L^2} + \|\mathcal{K}\|_{L^2 \to L^\infty}\|f-g\|_{L^2}\|g\|_{L^2}$$

where we used the fact that k is smooth and thus $\|\mathcal{K}\|_{L^1 \to L^\infty}, \|\mathcal{K}\|_{L^2 \to L^\infty}$ are bounded. It is easy to check that the linear potential part is also locally Lipschitz in L^2.

Thus we have that the entropy and the potential parts of the energy are locally Lipschitz in L^2, which finishes the proof of the lemma. $\square$

We prove now that the nonlinear Fokker-Planck equation evolves to the steady states in the H^1 norm.

Lemma 2 *For any nonnegative initial data of the nonlinear Fokker-Planck equation (5) satisfying (15) we have that the ω-limit set of the corresponding trajectory*

$$\Omega = \{\Psi; f(t_n) \xrightarrow{H^1} \Psi, \text{for some sequence } (t_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+\}$$

contains only steady states.

Proof. The compactness in $H^1 \cap \mathcal{C}$ of the semigroup generated by (5) (see *Remark* 3) together with the fact that all trajectories decay exponentially into a fixed ball suffice for having a connected global attractor in $H^1 \cap \mathcal{C}$ (see for instance [17], p.39, Thm. 3.4.6; note that there one has a semigroup defined on a Banach space. The fact that the semigroup is invariant with respect to a cone, as we have here, will not affect the validity of the quoted result, as one can easily check).

Take an arbitrary nonnegative initial data and consider the omega limit set associated to the trajectory starting from this initial data, Ω. Observe that all the elements in Ω have the same energy. Indeed, the energy is decreasing along the trajectories and it is bounded from below which means that there exists a $c \in \mathbb{R}$ so that $\lim_{t \to \infty} E(f(t)) = c$. We claim that this implies that the elements of Ω are the steady states.

Recall that the ω-limit set is an invariant set (see [17], p.36) so any trajectory starting from an initial data in Ω will stay in Ω and have the same energy c. For an initial data in Ω equation (16) implies $\log f - \mathcal{K}f - W = \text{const}, \forall t \geq 0$. Indeed, if the right hand side of (16) is negative at some time t_0 then it will be negative on an interval around t_0 and this would imply that $\mathcal{E}(t) < \mathcal{E}(0), \forall t > t_0$, which is a contradiction. From (5),(15) and $\log f - \mathcal{K}f - V = \text{const}, \forall t \geq 0$ we have that $f_t(t,\theta) = 0, \forall t \geq 0, \theta \in [0, 2\pi]$ i.e. $f(t,\theta) = f(0,\theta), \forall t \geq 0, \theta \in [0, 2\pi]$. $\square$

In general we do not know if Ω reduces to only one steady state. However, when we have a certain type of additional symmetry constraint then there are only finitely many steady states which can exist in Ω and only one of them will actually be in Ω for a given trajectory. We show this for the simple case when $k = \frac{1}{2}\cos(2\theta)$ so $\mathcal{K} = \mathcal{K}_{MS}$ and $V = 0$ hence (5) reduces to

$$\partial_t f + \partial_\theta(f\partial_\theta \mathcal{K}_{MS}f) = \partial_{\theta\theta}f \tag{17}$$

This equation has been analyzed in [5] where it was proved that if one starts with an even initial data then the evenness of the initial data will be preserved by the flow.

Therefore one can restrict oneself to studying solutions which have this symmetry, i.e. solutions of the form

$$f(t, \theta) = \frac{1}{2\pi} + \frac{1}{\pi} \Sigma_{k=1}^{\infty} y_k(t) \cos(2k\theta)$$

where $y_k(t) = \int_0^{2\pi} \cos(2k\theta) f(t, \theta) d\theta$.

The normalization of the initial data implies $y_0 = 1$ and $|y_k| \leq 1$. Then the nonlinear interaction potential becomes $\mathcal{K}_{MS}(\theta, t) = \frac{b}{2} y_1(t) \cos(2\theta)$.

In this setting the equation (17) can be rewritten in terms of Fourier coefficients as an infinite system of ODE's:

$$\begin{cases} y_0 = 1 \\ y_k' + 4k^2 y_k = bk y_1 (y_{k-1} - y_{k+1}), k = 1, 2, \ldots \end{cases}$$

For $k = 1$ we have

$$y_1' = y_1(-4 + by_1 - by_1 y_2)$$

which implies that if the $y_1(0) \neq 0$ then y_1 will be non-zero for all times.

It is known that there are only three *even around zero* possible steady states for the equation (17) (see [6]), an isotropic one $f_i(\theta) = \frac{1}{2\pi}$ and two nematic ones $f_{r(b)-}(\theta) = \frac{e^{-r(b)\cos(2\theta)}}{\int_0^{2\pi} e^{-r(b)\cos(2\theta)} d\theta}$ and $f_{r(b)+}(\theta) = \frac{e^{+r(b)\cos(2\theta)}}{\int_0^{2\pi} e^{+r(b)\cos(2\theta)} d\theta}$ where $r(b)$ depends on b. It is easy to check that $y_1(f_i) = 0$ and $y_1(f_{r(b)\pm}) \neq 0$. The eveness of initial data, which is propagated by the flow, leads thus to only three possible elements in the ω-limit set Ω. But Ω must be a connected set (see [17]) thus it will necessarily consist of only one element.

Summarizing the last observations, we have proved:

Lemma 3 *Consider the equation (17).*

If $y_1(0) \neq 0$ then $f(t, \theta)$ tends, as $t \to \infty$, to either $f_{r(b)-}$ or $f_{r(b)+}$. Otherwise, if $y_1(0) = 0$, we have $f(t, \theta) \to f_i$.

3 Homogeneous gradient flows

In this section we assume that the flow is externally imposed and of homogeneous gradient, that is:

$$u(x_1, x_2) = (\Sigma_{i=1}^2 u_i^1 x_i, \Sigma_{i=1}^2 u_i^2 x_i)$$

with $\{u_j^i\}_{i,j=1,2}$ arbitrary constants.

We consider the moving frame transformation

$$\bar{f}(t, x_1, x_2, \theta) = f(t, x_1 + t(\Sigma_{i=1}^2 u_i^1 x_i), x_2 + t(\Sigma_{i=1}^2 u_i^2 x_i), \theta)$$

and then the equation for f, (1), becomes an equation for $\bar{f}$

$$\partial_t \bar{f} + \partial_\theta [(\omega + s \cos(2\theta + \alpha)) \bar{f} + \partial_\theta \mathcal{K} \bar{f} \cdot \bar{f}] = \partial_{\theta\theta} \bar{f}$$

where we used the fact that V as given in (1) can be written $V = \omega + s \cos(2\theta + \alpha)$ with $\omega = \frac{u_1^2 - u_2^1}{2}$ the vorticity of the flow, $s > 0$ uniquely determined and α uniquely

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determined modulo 2π. The constants ω, s, α can be expressed in terms of $u^i_j, i, j = 1, 2$. One can easily check that $\bar{f}(t, \theta - \alpha/2)$ satisfies the equation

$$\partial_t \bar{f} + \partial_\theta[(\omega + s\cos(2\theta))\bar{f} + \partial_\theta \mathcal{K}\bar{f} \cdot \bar{f}] = \partial_{\theta\theta}\bar{f} \tag{18}$$

We continue working with the last equation. We start by observing that this is an equation of the form (5) and the assumptions of *Theorem* 1 are fulfilled. Thus equation (18) is dissipative.

Moreover, we can prove that for arbitrary ω, with small enough s and b the solutions evolve, in the long time limit, to a steady state. The strategy of the proof also shows that in the parameter regime given by assumptions (19),(20) below, there exists a unique steady state solution of (18).

Theorem 2 *Assume that $\omega \in \mathbb{R}$ and s, b are small enough so that*

$$1 > s + b\sqrt{2\pi}(\bar{C} + \frac{1}{\pi} + \epsilon)\|\partial_\theta k\|_{L^\infty} + \frac{b}{2}\|\partial_{\theta\theta}k\|_{L^\infty} \tag{19}$$

$$1 > 7s + \frac{b}{2}\|\partial_\theta^4 k\|_{L^\infty} + \frac{3}{2}b\|\partial_\theta^2 k\|_{L^\infty}$$

$$+b\sqrt{2\pi}\|k\|_{L^\infty}\left(\epsilon + (2s + b\|\partial_{\theta\theta}k\|_{L^\infty})(8\bar{C} + \frac{6}{\pi}) + \bar{C} + \frac{1}{\pi}\right) \tag{20}$$

for some $\epsilon > 0$ with $\bar{C}$ defined in Theorem 1.
Then any solution evolves to the unique steady state as $t \to \infty$.

Proof. We prove the statement in two steps. First we consider the difference between two arbitrary solutions and we show that after a certain time t_0 depending on the size of the initial data the two solutions will approach each other at an exponential rate. In the second step we use step 1 and a contraction argument to show that in fact any solution will have to evolve to a steady state.

Step 1 Consider the difference between two solutions f and g starting from the initial data $f(0)$ respectively $g(0)$:

$$\partial_t(f-g)+\omega\partial_\theta(f-g)+\partial_\theta[s\cos(2\theta)(f-g)+\partial_\theta\mathcal{K}(f-g)\cdot f+\partial_\theta\mathcal{K}g\cdot(f-g)] = \partial_{\theta\theta}(f-g) \tag{21}$$

Multiply by $f - g$, integrate over S^1 and by parts:

$$\frac{1}{2}\frac{d}{dt}\int_{S^1}(f-g)^2 - s\int_{S^1}\sin(2\theta)(f-g)^2(\theta)d\theta - \int_{S^1}\partial_\theta\mathcal{K}(f-g)\cdot f\partial_\theta(f-g)$$

$$+\frac{1}{2}\int_{S^1}\partial_{\theta\theta}(\mathcal{K}g)(f-g)^2 = -\int_{S^1}\partial_\theta(f-g)^2 \tag{22}$$

We have the following bound

$$|\int_{S^1}\partial_\theta\mathcal{K}(f-g)\cdot f\partial_\theta(f-g)| \leq \|\partial_\theta\mathcal{K}(f-g)\|_{L^\infty}\|f\|_{L^2}\|\partial_\theta(f-g)\|_{L^2}$$

$$\leq \|\partial_\theta\mathcal{K}\|_{L^2\to L^\infty}\|f-g\|_{L^2}R_1\|\partial_\theta(f-g)\|_{L^2} \leq b\sqrt{2\pi}\|\partial_\theta k\|_{L^\infty}R_1\|\partial_\theta(f-g)\|_{L^2}^2$$

where $R_1 = \bar{C} + \frac{1}{\pi} + \epsilon$ (see *Theorem* 1 for the definition of $\bar{C}$) and the above inequality holds after the time t_0 when the solution starting from initial data $f(0)$ enters the ball of radius R_1 in L^2 (this time t_0 depends only on the size of $f(0)$, see *Theorem* 1). For the last inequality we used the fact that $f - g$ is mean zero and Poincaré's inequality.

Also

$$\frac{1}{2}|\int_{S^1} \partial_{\theta\theta}(\mathcal{K}g)(f-g)^2| \leq \frac{1}{2}\|\partial_{\theta\theta}\mathcal{K}\|_{L^1 \to L^\infty}\underbrace{\|g\|_{L^1}}_{=1}\|f-g\|_{L^2}^2 \leq \frac{b}{2}\|\partial_{\theta\theta}k\|_{L^\infty}\|f-g\|_{L^2}$$

Using the above bounds in (22) we obtain, for $t \geq t_0$

$$\frac{1}{2}\frac{d}{dt}\|f-g\|_{L^2}^2 \leq -\|\partial_\theta(f-g)\|_{L^2}^2 + s\|f-g\|_{L^2}^2 + b\sqrt{2\pi}R_1\|\partial_\theta k\|_{L^\infty}\|\partial_\theta(f-g)\|_{L^2}^2$$
$$+\frac{b}{2}\|\partial_{\theta\theta}k\|_{L^\infty}\|f-g\|_{L^2}$$

and by assumption (19) and Poincaré's inequality we have that the difference between f and g will decay exponentially after time t_0.

In order to evaluate the difference in the H^1 norm we take the derivative of the equation (18) with respect to θ, for two solutions f and g. We denote $\partial_\theta f = F, \partial_\theta g = G$ and then we have an equation for $F - G$:

$$\partial_t(F-G) + \partial_\theta[(F-G)(\omega + s\cos(2\theta)) - 2s\sin(2\theta)(f-g)$$
$$+(F-G)\partial_\theta\mathcal{K}g + F\partial_\theta\mathcal{K}(f-g)$$
$$+(f-g)\partial_{\theta\theta}(\mathcal{K}g) + f\partial_{\theta\theta}\mathcal{K}(f-g)] = \partial_{\theta\theta}(F-G)$$

Multiplying by $F - G$, integrating over S^1 and by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{S^1}(F-G)^2 - \underbrace{s\int_{S^1}\sin(2\theta)(F-G)^2}_{T_1}$$

$$-\underbrace{4s\int_{S^1}\cos(2\theta)(f-g)(F-G)}_{T_2} - \underbrace{2s\int_{S^1}\sin(2\theta)(F-G)^2}_{T_3}$$

$$+\underbrace{\frac{1}{2}\int_{S^1}\partial_{\theta\theta}(\mathcal{K}g)(F-G)^2}_{T_4} - \underbrace{\int_{S^1}F\mathcal{K}(F-G)\partial_\theta(F-G)}_{T_5}$$

$$+\underbrace{\int_{S^1}(F-G)^2\partial_{\theta\theta}(\mathcal{K}g)}_{T_6} + \underbrace{\int_{S^1}(f-g)\partial_\theta^3(\mathcal{K}g)(F-G)}_{T_7}$$

$$-\underbrace{\int_{S^1}f\partial_\theta\mathcal{K}(F-G)\partial_\theta(F-G)}_{T_8} = -\int_{S^1}\partial_\theta(F-G)^2 \qquad (23)$$

where we used (4) on the last line three lines to commute ∂_θ and $\mathcal{K}$.

We bound each term $T_i, i = 1, \ldots, 8$ by $c_i \|F - G\|_{L^2}^2$ or $c_i \|\partial_\theta(F - G)\|_{L^2}^2$ with some appropriate constants c_i.

$$|T_1 + T_3 + T_4 + T_6| \leq (s + 2s + \frac{3}{2} \cdot b \|\partial_{\theta\theta} k\|_{L^\infty}) \|F - G\|_{L^2}^2 \tag{24}$$

We bound T_2 and T_7 in the same manner. We show only how to bound T_2. Using an integration by parts twice we have

$$|4s \int_{S^1} \cos(2\theta)(f - g)\partial_\theta(f - g)| = |4s \int_{S^1} \sin(2\theta)(f - g)^2|$$
$$\leq 4s \|f - g\|_{L^2}^2 \leq 4s \|F - G\|_{L^2}^2$$

where we used Poincaré's inequality in the last relation.

Then

$$|T_2 + T_7| \leq (4s + \frac{b}{2} \|\partial_\theta^4 k\|_{L^\infty}) \|F - G\|_{L^2}^2 \tag{25}$$

The terms T_5 and T_8 are treated in the same manner. We show again just how to bound one of them

$$|T_5| \leq \|F\|_{L^2} \|\mathcal{K}(F - G)\|_{L^\infty} \|\partial_\theta(F - G)\|_{L^2}$$
$$\leq R_2 \|\mathcal{K}\|_{L^2 \to L^\infty} \|F - G\|_{L^2} \|\partial_\theta(F - G)\|_{L^2} \leq R_2 b \|k\|_{L^\infty} \sqrt{2\pi} \|\partial_\theta(F - G)\|_{L^2}^2$$

where we used Poincaré's inequality in the last relation with the constant $R_2 = \epsilon + (2s + b \|\partial_{\theta\theta} k\|_{L^\infty})(8\bar{C} + \frac{6}{\pi})$, for some $\epsilon > 0$ and the relation holds for the time $t \geq t_1$ after which $F = \partial_\theta f \in B_{L^2}(0, R_2)$ (see *Theorem 1*).

Hence

$$|T_5 + T_8| \leq (R_2 + R_1) b \|k\|_{L^\infty} \sqrt{2\pi} \|\partial_\theta(F - G)\|_{L^2}^2 \tag{26}$$

where the inequality holds for $t \geq t_2 = \max\{t_1, t_0\}$, where t_2 is the time after which $f \in B(0, R_1), F = \partial_\theta f \in B(0, R_2)$ (see *Theorem 1*).

Using bounds (24),(25),(26) into (23) together with Poincaré's inequality and assumption (20) we obtain that $\|F - G\|_{L^2}^2$ decays exponentially after time t_2.

Step 2 Consider the ball $B(0, R)$ in H^1 for some $R > \max\{R_1, R_2\}$. Then *Theorem 1* and the first step show that there exists a time t_3 such that for all $t > t_3$ we have

$$S(t) : B(0, R) \cap \mathcal{C} \cap S_{L^1}(0, 1) \to B(0, R) \cap \mathcal{C} \cap S_{L^1}(0, 1) \tag{27}$$

$$\|S(t)f_0 - S(t)g_0\|_{H^1} \leq \alpha \|f_0 - g_0\|_{H^1} \tag{28}$$

for some $\alpha < 1$, where $S(t)$ denotes the nonlinear semigroup generated by the equation. We denoted by $\mathcal{C}$ the cone of nonnegative functions and by $S_{L^1}(0, 1)$ the sphere of radius 1, centered at 0, in L^1.

Let $X = B(0, R) \cap \mathcal{C} \cap S_{L^1}(0, 1)$. Then X with the metric induced by the H^1 norm is a complete metric space. Define

$$T : X \to X, \ Tf = S(a)f$$

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for some $a \in \mathbb{Q}, a > t_3$. The previous arguments show that T thus defined is a contraction.

Denote $T \circ T \circ \cdots \circ T = T^n$. As T is a contraction we have that as $n \to \infty$, $T^n f_0 \to f_1$ where f_1 is a fixed point of T.

Similarly, taking $Uf = S(b)f$ for $b \in \mathbb{R} - \mathbb{Q}, b > t_3$, we have $U : X \to X$ is a contraction. Reasoning as before we obtain the existence of a $f_2 \in X$ such that $Uf_2 = f_2$.

From *Step 1* we have

$$\lim_{n \to \infty} \|S(na)f_1 - S(na)f_2\|_{H^1} = 0$$

But $S(na)f_1 = T^n f_1 = f_1$ so the last limit becomes:

$$\|f_1 - S(na)f_2\|_{H^1} \to 0, \text{ as } n \to \infty \tag{29}$$

Recall Hurwitz's theorem in number theory (see for instance [30]) which states that for $\gamma \in \mathbb{R} - \mathbb{Q}$ there are infinitely many rationals $\frac{p}{q}$ such that

$$|\gamma - \frac{p}{q}| < \frac{1}{\sqrt{5}q^2}$$

An easy consequence is that there exist two sequences $(m_k)_{k \in \mathbb{N}}, (n_k)_{k \in \mathbb{N}}; m_k, n_k \geq k, \forall k \in \mathbb{N}$ such that

$$|n_k a - m_k b| < \frac{1}{k}$$

Let $\epsilon > 0$. As $S(t)f_2$ is continuous at $t = 0$ there exists a $\delta > 0$ such that

$$|t| < \delta \Rightarrow \|S(t)f_2 - f_2\|_{H^1} < \frac{\epsilon}{2} \tag{30}$$

Thus, for k large enough so that $|n_k a - m_k b| < \frac{1}{k} < \delta$ we have

$$\|S(n_k a)f_2 - S(m_k b)f_2\|_{H^1} = \|S(n_k a - m_k b)S(m_k b)f_2 - S(m_k b)f_2\|_{H^1}$$
$$= \|S(n_k a - m_k b)f_2 - f_2\|_{H^1} < \frac{\epsilon}{2} \tag{31}$$

where for the first equality we used the semigroup property and for the last inequality relation (30).

On the other hand from (29) we know that for there exists a rank n_0 such that for $n_k \geq n_0$

$$\|f_1 - S(n_k a)f_2\|_{H^1} \leq \frac{\epsilon}{2} \tag{32}$$

Putting together (31) and (32) we obtain

$$\|f_1 - f_2\|_{H^1} < \epsilon$$

and since ϵ is arbitrary we have that $f_1 = f_2$.

We show now that $S(t)f_1$ is periodic of arbitrarily small period.

Take m_k, n_k such that $|n_k a - m_k b| < \frac{1}{k}$. Assuming without loss of generality that $n_k a < m_k b$ we have

$$S(m_k b - n_k a)f_1 = S(m_k b - n_k a)S(n_k a)f_1 = S(m_k b)f_1 = S(m_k b)f_2 = f_2 = f_1$$

and thus $S(t)f_1$ has time period $0 < m_k b - n_k a < \frac{1}{k}$. It is well known that a continuous function of arbitrarily small periods must be constant. Thus f_1 is a steady state.

As the difference of any two solutions tends to 0 as $t \to \infty$ this shows that all solutions tend to the steady state f_1, which must be unique. $\square$

Following the proof, one can easily see that we also have

Corollary 1 *If (19) and (20) are satisfied there exists a unique steady state solution of the equation (18), for an arbitrary smooth potential k satisfying (4).*

Let us observe now that the presence of the flow introduces in the equation a term of the form $\partial_\theta[(\omega + s\cos(2\theta))\bar{f})]$. The parameters ω and s play very different rôles. In the case when either ω or s is zero we have that the equation still has a gradient structure, in an appropriate reference frame.

Lemma 4 *Consider equation (18). The following holds:*
(i) If $\omega = 0, s \neq 0$ equation (18) is of gradient type and the ω-limit set of any solution consists of steady states solutions of (18).
(ii) If $s = 0, \omega \neq 0$ make the rotating frame transformation

$$\tilde{f}(t,\theta) = \bar{f}(t,\theta + \omega t)$$

Then $\tilde{f}$ satisfies the equation (5) (with $V = 0$) which is an equation of gradient type.
Moreover, we have a time periodic solution (in the moving frame) of (18), namely $g(\theta - \omega t)$ where g is a steady state solution of (18) for $\omega = s = 0$.
(iii) (an isotropic-nematic pattern) Assume that $s = 0, \omega \neq 0$. Consider solutions of (18) with Maier-Saupe potential, for which the initial data is even around 0, in the θ variable. As $t \to \infty$ we have

$$\bar{f}(t, x_1, x_2, \theta) \overset{H^1}{\to} g\,(\theta - \omega t)$$

where $g = \frac{1}{2\pi}$ if $y_1(f(0,\cdot)) = 0$ and $g \in \{f_{r(b)}, f_{-r(b)}\}$ if $y_1(\bar{f}(0,\cdot)) \neq 0$ (with $r(b)$ a function of b, see [6], Lemma 3 and $y_1(\bar{f}(0,\cdot)) = \int_0^{2\pi} \bar{f}(0, x_1, x_2, \theta)\cos(2\theta)d\theta$).

Proof. (i) In this case condition (15) is satisfied, as $V(\theta) = \cos(2\theta) = \frac{d}{d\theta}\frac{\sin(2\theta)}{2}$. Lemma 2 gives us the conclusion.

(ii) Observe that $\mathcal{K}$ is invariant under the rotating frame transformation

$$
\begin{aligned}
(\mathcal{K}\bar{f})(\theta + \omega t) &= \int_{S^1} k(\theta + \omega t - \theta')\bar{f}(\theta')d\theta' \\
&= \int_{S^1} k(\theta + \omega t - \theta' - \omega t)\bar{f}(\theta' + \omega t)d\theta' = (\mathcal{K}\tilde{f})(\theta) \quad (33)
\end{aligned}
$$

thus $\tilde{f}$ satisfies the equation

$$\partial_t \tilde{f} + \partial_\theta(\tilde{f}\partial_\theta(\mathcal{K}\tilde{f})) = \partial_{\theta\theta}\tilde{f} \quad (34)$$

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which is of gradient type.

A direct computation, using (33), shows that $g(\theta - \omega t)$ is a solution of (18), when $s = 0$ (where g is a steady state solution of (34)).

(iii) Consider the same rotating frame transformation as in the previous part. Using the fact that the initial data is the same in the moving frame as in the fixed frame and taking into account *Lemma* 3 we are done. $\square$

The last lemma suggests that one can think, heuristically, that ω affects the rotational behavior of the solution while s affects the long time shape of the solution. The rôles of ω and s are thus very different. When they are both non-zero one can not reduce the equation to a gradient type one just by considering it in the rotating frame. What prevents us from repeating the argument above is the fact that in a moving frame the linear potential V becomes time dependent. In this situation there is no obvious Lyapunov functional.

Nevertheless, when both s and ω are nonzero, we can prove that for s small enough one has time periodic solutions for equation (18). Taking into account that (18) is equation (1) in a moving frame, this implies that in the initial frame we have tumbling wave solutions.

The following argument is done just for the Maier-Saupe potential. This is because at the present time only for this potential there is a good understanding of the form of the steady state solutions (in the absence of the flow) and of their dependence on the concentration intensity parameter b.

Theorem 3 *Consider equation (18) with $\omega \neq 0$ and let $\mathcal{K} = \mathcal{K}_{MS}$ be the Maier-Saupe potential. For $b > 4$ and arbitrary $0 < W_1 < W_2$ there is an S depending on W_1 and W_2 such that for $s \in [0, S], s < \frac{1}{2}(W_2 - W_1)$ and $|\omega| \in [W_1 + s, W_2 - s]$ the equation (18) has a time periodic solution.*

Proof. We present first the strategy of the proof. We are interested in obtaining zeroes of a functional $\mathcal{F} : \mathcal{X} \times \mathbb{R} \times \mathbb{R} \to \mathcal{Y}$

$$\mathcal{F}(f, \omega, s) = \partial_t f + \partial_\theta \left[(\omega + s \cos(2\theta)) f \right] + \partial_\theta \left[\partial_\theta (\mathcal{K} f) \cdot f \right] - \partial_{\theta\theta} f$$

where $\mathcal{X}, \mathcal{Y}$ are spaces of functions periodic both in t and θ, whose precise definition will not be given because we will see soon that it is more convenient to work with a different formulation of the above functional. For that formulation we will make precise the functional spaces.

We will show that for any ω there exists a λ_ω such that for any $\lambda \in (-\lambda_\omega, \lambda_\omega)$ we have

$$\mathcal{F}(f, \omega + \lambda, \lambda) = 0 \tag{35}$$

for some $f \in \mathcal{X}$. This suffices for obtaining the conclusion of the theorem. Indeed, this shows that for arbitrary ω there exists an open interval around ω, namely $(\omega - \lambda_\omega, \omega + \lambda_\omega)$ such that for any $\mu \in (\omega - \lambda_\omega, \omega + \lambda_\omega), \nu \in (-\lambda_\omega, \lambda_\omega), \mu - \nu = \omega$ we have $\mathcal{F}(f, \mu, \nu) = 0$ for some $f \in \mathcal{X}$. The set $\{x : W_1 \leq |x| \leq W_2\}$ is compact so there exists a finite covering with intervals of the form $(\omega - \lambda_\omega, \omega + \lambda_\omega)$, say

$$\{x : W_1 \leq |x| \leq W_2\} = \cup_{k=1}^{l} (\omega_k - \lambda_{\omega_k}, \omega_k + \lambda_{\omega_k})$$

We take now $S = \min_{k \in \{1,...,l\}} \lambda_{\omega_k}$ and we obtain the conclusion.

Returning to (35) let us consider the Smoluchowski equation in homogeneous flow

$$\partial_t f + \partial_\theta[(\omega + \lambda + \lambda\cos(2\theta))f + \partial_\theta(\mathcal{K}f)f] = \partial_{\theta\theta}f \qquad (36)$$

Make the rotating frame transformation $\tilde{f}(t,\theta) = f(t,\theta + \omega t)$. Then (36) becomes

$$\partial_t\tilde{f} + \lambda\partial_\theta[\tilde{f} + \cos(2\theta + 2\omega t)\tilde{f}] + \partial_\theta[\partial_\theta(\mathcal{K}\tilde{f}) \cdot \tilde{f}] = \partial_{\theta\theta}\tilde{f} \qquad (37)$$

Let $g(\theta)$ be an even, *nonconstant* solution of

$$\partial_\theta[\partial_\theta(\mathcal{K}f)f] = \partial_{\theta\theta}f$$

We know that such a solution exists and it is given by the formula

$$g = \frac{e^{r(b)\cos(2\theta)}}{\int_0^{2\pi} e^{r(b)\cos(2\theta)}\,d\theta}$$

for a certain $r(b)$ satisfying a nonlinear equation (see [6]).

We decompose $\tilde{f}(t,\theta) = z(t,\theta) + g$. Then z satisfies the equation

$$\partial_t z + \lambda\partial_\theta[(z+g) + \cos(2\theta + 2\omega t)(z+g)] + \partial_\theta[\partial_\theta(\mathcal{K}z)g + \partial_\theta(\mathcal{K}g)z + \partial_\theta(\mathcal{K}z)z] = z_{\theta\theta}$$

We prove the existence of time periodic solutions for the above equation, with time period $\frac{\pi}{\omega}$. Define $F : \mathcal{X} \times \mathbb{R} \to \mathcal{Y}$

$$\begin{aligned} F(z,\lambda) = {} & \partial_t z + \lambda\partial_\theta[(z+g) + \cos(2\theta + 2\omega t)(z+g)] \\ & + \partial_\theta[\partial_\theta(\mathcal{K}z)g + \partial_\theta(\mathcal{K}g)z + \partial_\theta(\mathcal{K}z)z] - z_{\theta\theta} \end{aligned} \qquad (38)$$

with

$$\begin{aligned} \mathcal{X} = \{ & z \in H^1\left([0,\tfrac{\pi}{\omega}], H^2[0,2\pi]\right), z(t,0) = z(t,2\pi), \forall t \in [0,\tfrac{\pi}{\omega}], \\ & z(0,\theta) = z(\tfrac{\pi}{\omega},\theta), \partial_t z(0,\theta) = \partial_t z(\tfrac{\pi}{\omega},\theta,)\forall\theta \in [0,2\pi], \\ & \int_0^{2\pi} z(t,\theta)d\theta = 0, \forall t \in [0,\tfrac{\pi}{\omega}]\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{Y} = \{ & z \in L^2\left([0,\tfrac{\pi}{\omega}], L^2[0,2\pi]\right), z(0,\theta) = z(\tfrac{\pi}{\omega},\theta), \forall\theta \in [0,2\pi], \\ & z(t,0) = z(t,2\pi), \forall t \in [0,\tfrac{\pi}{\omega}], \int_0^{2\pi} z(t,\theta)d\theta = 0, \forall t \in [0,\tfrac{\pi}{\omega}]\} \end{aligned}$$

Remark 4 *The choice of regularity spaces is somewhat arbitrary, as one can see a posteriori that the solution is analytic in time and space. What we need for our proof is that the norm of $\mathcal{X}$ controls the L^∞ norm in time and space. We also need that the structure of $\mathcal{Y}$ allows for a simple orthogonal decomposition.*

We have then that

$$F(0,0) = 0$$

We want to apply the implicit function theorem and obtain the existence of a periodic solution for small λ. This is a continuation argument, which finds a periodic solution of period $\frac{\pi}{\omega}$ near one which we already know to exists (for $\lambda = 0$, see *Lemma 4*). It is due to the fact that we are working in a rotating frame that the time periodic solution in the initial frame, $g(\theta - \omega t)$, is stationary in the rotating frame, g in (38).

In order to apply the implicit function theorem we need thus to check that

$$Lh = \partial_z F(0,0)h = \partial_t h + \partial_\theta [\partial_\theta \mathcal{K}g \cdot h + \partial_\theta \mathcal{K}h \cdot g] - \partial_{\theta\theta} h$$

as a bounded operator from $\mathcal{X}$ to $\mathcal{Y}$ is a homeomorphism, i.e., taking into account the open mapping theorem, that L is bijective. Nevertheless, this is not the case since, as we will see, $dim(ker(L)) = dim(range(L)) = 1$ so L is a an operator of Fredholm index 0. In this situation an implicit function theorem is still possible under a certain *"non-resonance"* condition. This is available for instance in [21], p.12. We will present it in *Lemma 6* below after analyzing the operator L.

In order to determine the kernel and the range of the operator L we need to study equations of the form $Lh = f$, for $f \in \mathcal{Y}$. Multiplying such an equation by $\sqrt{\frac{\omega}{\pi}} e^{-ik2\omega t}$ and integrating on $[0, \frac{\pi}{\omega}]$ we have that the equation $Lh = f$ reduces to a decoupled system of ordinary differential equations:

$$i2\omega k h_k + \partial_\theta [\partial_\theta (\mathcal{K}g)h_k + \partial_\theta (\mathcal{K}h_k)g] - \partial_{\theta\theta} h_k = f_k$$

where we denote by h_k, f_k the k-th Fourier mode in time of h, respectively f.

Thus the problem of determining the kernel and range of L reduces to understanding the operator

$$\tilde{L}(h) = \partial_{\theta\theta} h - \partial_\theta [\partial_\theta (\mathcal{K}g)h + \partial_\theta (\mathcal{K}h)g] \tag{39}$$

We have

Lemma 5 *Let $\tilde{L} : \tilde{X} \to \tilde{Y}$ be a bounded operator as defined in (39) with*

$$\tilde{X} = \{f \in H^2[0, 2\pi], \int_0^{2\pi} f(\theta)d\theta = 0\}, \ \tilde{Y} = \{f \in L^2[0, 2\pi], \int_0^{2\pi} f(\theta)d\theta = 0\}.$$

Then

$$\ker(\tilde{L}) = \{p\partial_\theta g, p \in \mathbb{R}\}$$

and

$$f \in Range(\tilde{L}) \Leftrightarrow \int_0^{2\pi} (\int_0^\theta f(\sigma)d\sigma)(\frac{1}{g(\theta)} - \frac{1}{2\pi} \int_0^{2\pi} \frac{d\sigma}{g(\sigma)})d\theta = 0 \tag{40}$$

Thus $dim(Ker(\tilde{L})) = codim(Range(\tilde{L})) = 1$, so $\tilde{L}$ is an operator of Fredholm index 0.

Moreover, regarding $\tilde{L}$ as an unbounded operator on $\tilde{Y}$, with $D(\tilde{L}) = \tilde{X}$, we have that $\tilde{L}$ is a sectorial operator with discrete spectrum contained in the real line.

Proof. Standard arguments show that the operator, regarded as an unbounded operators on $\tilde{Y}$, has discrete spectrum and it is sectorial (see for instance [18],[20]). The proof will not be given here.

Let us observe that the operator can only have real spectrum, i.e. only real eigenvalues.

Define

$$Ah = \frac{h}{g} - Kh \tag{41}$$

and then

$$\tilde{L}h = (h_\theta - h\frac{\partial_\theta g}{g} - g(Kh)_\theta)_\theta + (\frac{h}{g}\underbrace{[\partial_\theta g - g(Kg)_\theta]}_{=0})_\theta = (g(Ah)_\theta)_\theta \tag{42}$$

where the cancellation is due to our choice of g.

In order to compute the spectrum of $\tilde{L}$ consider the equation

$$\tilde{L}(h_R + ih_I) = (R + iI)(h_R + ih_I)$$

with R, I, h_R, h_I real quantities.

Separating the real and imaginary parts in the above equation we obtain

$$(g(Ah_R)_\theta)_\theta = Rh_R - Ih_I \tag{43}$$

$$(g(Ah_I)_\theta)_\theta = Ih_R + Rh_I \tag{44}$$

Multiplying (43) by Ah_R, adding the result to (44) multiplied by Ah_I, integrating over $[0, 2\pi]$ and by parts we have

$$-\int_0^{2\pi} g((Ah_R)_\theta)^2 - \int_0^{2\pi} g((Ah_I)_\theta)^2 = R(Ah_R, h_R) + R(Ah_I, h_I) \tag{45}$$

where we used the fact that

$$\int_0^{2\pi} h_I Ah_R = \int_0^{2\pi} h_R Ah_I \tag{46}$$

Also, let us observe that by multiplying (43) by Ah_I, integrating over $[0, 2\pi]$ and by parts, we obtain on the left hand side of the equality the same thing as multiplying (44) by Ah_R, integrating over $[0, 2\pi]$ and by parts. This implies the equality of the corresponding right hand sides, i.e.

$$\int_0^{2\pi} Rh_R Ah_I - \int_0^{2\pi} Ih_I Ah_I = \int_0^{2\pi} Ih_R Ah_R + \int_0^{2\pi} Rh_I Ah_R \tag{47}$$

and using again (46) we have

$$-I(Ah_I, h_I) = I(h_R, Ah_R) \tag{48}$$

which, for $I \neq 0$, used in (45) implies

$$-\int_0^{2\pi} g((Ah_R)_\theta)^2 - \int_0^{2\pi} g((Ah_I)_\theta)^2 = 0$$

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Taking into account that $g > 0$ the last equality implies that $(Ah_I)_\theta = (Ah_R)_\theta \equiv 0$. Using this into (43),(44) we obtain $h_I = h_R = 0$. Thus necessarily $I = 0$, so the imaginary part of an eigenvalue must be zero.

In order to compute the range and the kernel, take $f \in \tilde{Y}$ and denote $F(\theta) = \int_0^\theta f(\sigma)d\sigma$. Then

$$\tilde{L}h = \partial_\theta(g(Ah)_\theta) = \partial_\theta F$$

implies

$$g(Ah)_\theta = F(\theta) + c_1 \tag{49}$$

In order to determine c_1 we divide by g on both sides of the last equality and integrate on $[0, 2\pi]$ obtaining

$$c_1 = -\frac{\int_0^{2\pi} \frac{F(\theta)}{g(\theta)}d\theta}{\int_0^{2\pi} \frac{1}{g(\theta)}d\theta} \tag{50}$$

Also, integrating on $[0, 2\pi]$ both sides of (49) we get

$$\int_0^{2\pi} g(Ah)_\theta = \int_0^{2\pi} F(\theta)d\theta + c_1 \cdot 2\pi \tag{51}$$

On the other hand

$$\int_0^{2\pi} g(Ah)_\theta = -\int_0^{2\pi} g_\theta \cdot Ah$$

$$= 2r(b)\int_0^{2\pi} g(\theta)\sin(2\theta)[\frac{h}{g} - \frac{b}{2}c(h)\cos(2\theta) - \frac{b}{2}s(h)\sin(2\theta)]d\theta$$

$$= 2r(b)[s(h) - \frac{b}{2}(\frac{1}{2} - \frac{1}{2}(1 - \frac{4}{b}))s(h)] = 0$$

where we used in the first equality an integration by parts and in the second the relation $g_\theta = -2r(b)g\sin(2\theta)$ (see the definition of g). Also we denoted $c(h) = \int_0^{2\pi} h(\theta)\cos(2\theta)d\theta$, $s(h) = \int_0^{2\pi} h(\theta)\sin(2\theta)d\theta$ and we used the fact that $\int_0^{2\pi} g(\theta)\cos(4\theta)d\theta = 1 - \frac{4}{b}$ (see [3]). Using the last computation in (51) we obtain

$$2\pi c_1 = -\int_0^{2\pi} F(\theta)d\theta \tag{52}$$

From (50) and (52) we have that c_1 is equal to both sides of the equality

$$-\frac{\int_0^{2\pi} \frac{F(\theta)}{g(\theta)}d\theta}{\int_0^{2\pi} \frac{1}{g(\theta)}d\theta} = -\frac{\int_0^{2\pi} F(\theta)d\theta}{2\pi}$$

which implies a restriction on F namely (40).

Observe that relation (40) is the only restriction on the range of $\tilde{L}$. Indeed, returning to (49) and dividing by g, integrating and recalling the definition of Ah, (41) we have

$$h(\theta) = \frac{b}{2}c(h)g(\theta)\cos(2\theta) + \frac{b}{2}s(h)g(\theta)\sin(2\theta) + g(\theta)\int_0^\theta \frac{F(\sigma) + c_1}{g(\sigma)}d\sigma + c_2 g(\theta) \tag{53}$$

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where c_2 is a constant of integration to be determined.

Multiplying (53) by $\cos(2\theta)$ and integrating over $[0, 2\pi]$ we obtain

$$c(h) = \frac{b}{2}c(h)[\frac{1}{2} + \frac{1}{2}(1 - \frac{4}{b})] + \int_0^{2\pi} g(\theta)\cos(2\theta)\left(\int_0^\theta \frac{F(\sigma) + c_1}{g(\sigma)}d\sigma\right)d\theta + c_2 c(g)$$

which implies

$$c(h)(2 - \frac{b}{2}) = \int_0^{2\pi} g(\theta)\cos(2\theta)\left(\int_0^\theta \frac{F(\sigma) + c_1}{g(\sigma)}d\sigma\right)d\theta + c_2 c(g) \tag{54}$$

Also, integrating (53) over $[0, 2\pi]$ and using the fact that we are looking for solutions h which have mean zero we get

$$-c_2 = \frac{b}{2}c(h)c(g) + \int_0^{2\pi} g(\theta)\left(\int_0^\theta \frac{F(\sigma) + c_1}{g(\sigma)}d\sigma\right)d\theta \tag{55}$$

Multiplying the last relation by $-c(g)$ and replacing the expression for $c_2 c(g)$ thus obtained into (54) we have

$$c(h)(2 - \frac{b}{2} + \frac{b}{2}c(g)^2) = -c(g)\int_0^{2\pi} g(\theta)\left(\int_0^\theta \frac{F(\sigma) + c_1}{g(\sigma)}d\sigma\right)d\theta$$

$$+ \int_0^{2\pi} g(\theta)\cos(2\theta)\left(\int_0^\theta \frac{F(\sigma) + c_1}{g(\sigma)}d\sigma\right)d\theta \tag{56}$$

In the last relation we can divide by $2 - \frac{b}{2} + \frac{b}{2}c^2(g)$ (which is nonzero as $c^2(g) > 1 - \frac{4}{b}$, for $b > 4$ see [24], *Theorem* 2.1) and thus we obtain an expression for $c(h)$ only in terms of F and g (as c_1 can also be determined in terms of F and g, see (50)).

Let us check that we can take $s(h)$ to be arbitrary in the representation formula for h. Indeed, multiplying (53) by $\sin(2\theta)$ and integrating over $[0, 2\pi]$ we get

$$s(h) = \frac{b}{2}s(h)\underbrace{[\frac{1}{2} - \frac{1}{2}(1 - \frac{4}{b})]}_{2/b} + \int_0^{2\pi} g(\theta)\sin(2\theta)(\int_0^\theta \frac{F(\sigma) + c_1}{g(\sigma)}d\sigma)d\theta$$

which is always true as

$$\int_0^{2\pi} \underbrace{g(\theta)\sin(2\theta)}_{=-g_\theta/(2r(b))}(\int_0^\theta \frac{F(\sigma) + c_1}{g(\sigma)}d\sigma)d\theta = -\frac{1}{2r(b)}g(2\pi)\underbrace{\int_0^{2\pi}(\frac{F(\sigma) + c_1}{g(\sigma)})d\sigma}_{=0}$$

$$+ \frac{1}{2r(b)}\underbrace{\int_0^{2\pi} g(\theta)\frac{F(\theta) + c_1}{g(\theta)}d\theta}_{=0}$$

where we used an integration by parts for the equality and (50),(52) for the cancelations.

Summarizing: for a given $f \in \tilde{L}$, satisfying the compatibility condition (40) we define c_1 by relation (50) and use this in (56) to obtain an expression for $c(h)$. We use this

to get c_2 from (55) and plug everything in (53). The h thus obtained will satisfy the equation $\tilde{L}h = f$ for an arbitrary $s(h)$.

In particular, if we look for a solution of $\tilde{L}h = 0$ we obtain that $h \in \{p\partial_\theta g, p \in \mathbb{R}\}$ which gives us the kernel of $\tilde{L}$. $\square$

The properties of $\tilde{L}$ imply that the full operator L has the same kernel as $\tilde{L}$(where now $p\partial_\theta g, p \in \mathbb{R}$ is regarded as an element in $\mathcal{X}$). Also, the compatibility condition for $f \in \mathcal{Y}$ to be in the range of L is

$$\int_0^{\pi/(\omega)} \int_0^{2\pi} (\int_0^\theta f(\sigma, s)d\sigma)(\frac{1}{g(\theta)} - \frac{1}{2\pi} \int_0^{2\pi} \frac{d\sigma}{g(\sigma)})d\theta ds = 0 \qquad (57)$$

so $range(L)$ has codimension one.

A technical remark is necessary at this point: we have that for $f \in \mathcal{Y}$ satisfying the appropriate compatibility condition as above there exists some h such that $Lh = f$. In order to make sure that this h is in $\mathcal{X}$ we need to check the regularity in time. While our argument so far does not give but L^2 regularity in time, higher regularity in time is nevertheless available. This is a consequence of the parabolic nature of the operator L.

More precisely, the equation

$$ik2\omega h_k + \tilde{L}h_k = f_k$$

and the fact that $ik2\omega$ is in the resolvent of $\tilde{L}$, imply that

$$h_k = (ik2\omega + \tilde{L})^{-1}f_k, \forall k \in \mathbb{Z} - \{0\}$$

with

$$\|h_k\|_{H^2} \le \|(ik2\omega + \tilde{L})^{-1}\|_{L^2 \to H^2}\|f_k\|_{L^2} \qquad (58)$$

If $k = 0$ we have (by the compatibility condition (57)) $f_0 \in range(\tilde{L})$ and thus

$$h_0 = \tilde{L}^{-1}f_0$$

with

$$\|h_0\|_{H^2} \le \|\tilde{L}^{-1}\|_{L^2 \to H^2}\|f_0\|_{L^2} \qquad (59)$$

As $\tilde{L}$ is sectorial we have that for $|k|$ large enough $|k| > k_0 > 0$ the number $ik2\omega$ is in the resolvent (which we already knew from the lemma) and moreover (see [15])

$$\|(ik2\omega + \tilde{L})^{-1}\|_{L^2 \to H^2} \le C, \forall |k| > k_0 > 0 \qquad (60)$$

where C is a constant independent of k.

Let $\tilde{C} = \max\{C, \|(ik2\omega + \tilde{L})^{-1}\|_{L^2 \to H^2}, k = 0, \pm 1, \pm 2, \dots, \pm k_0\}$. We have then

$$\|h_k\|_{H^2} \le \tilde{C}\|f_k\|, \forall k \in \mathbb{Z}$$

which implies that $h \in L^2((0, \frac{\pi}{\omega}), H^2[0, 2\pi])$, i.e. the same regularity in time as f. Also, using the equation $Lh = \partial_t h + \partial_\theta[\partial_\theta \mathcal{K}g \cdot h + \partial_\theta \mathcal{K}h \cdot g] - \partial_{\theta\theta}h = f$ we get $\partial_t h \in L^2((0, \frac{\pi}{\omega}), L^2(0, 2\pi))$ so $h \in C([0, \frac{\pi}{\omega}], L^2(0, 2\pi))$ (see for instance [9]).

On the other hand we also have that h is a weak solution of a Cauchy problem for the equation $Lh = f$, with initial data $h(0) \in L^2(0, 2\pi)$. But $h(0) = h(\frac{\pi}{\omega})$ by time

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periodicity and then the parabolic regularization effect implies $h(0) = h(\frac{\pi}{\omega}) \in H^2[0, 2\pi]$. For $f \in L^2(0, \frac{2\pi}{\omega}, L^2(0, 2\pi))$ the Cauchy problem with initial data $h(0) \in H^2[0, 2\pi]$ has a unique solution $g \in H^1([0, \frac{2\pi}{\omega}], H^2[0, 2\pi])$ with $g(0) = h(0)$ (see [15]). Moreover it can be shown that the uniqueness holds for solutions which are only in $C([0, \frac{\pi}{\omega}], L^2(0, 2\pi))$. Then $g \equiv h$ and thus h will have the necessary regularity in time for being in $\mathcal{X}$.

The abstract lemma (which extends the Implicit Function Theorem) that we need is

Lemma 6 *([21], p.12) Let $F : U \times V \to Z$ with $U \subset X, V \subset \mathbb{R}$, where X and Z are Banach spaces. Assume that $F \in C^1(U \times V, Z)$ and:*

- *$F(x_0, \lambda_0) = 0$ for some $(x_0, \lambda_0) \in U \times V$, $Range(D_x F(x_0, \lambda_0))$ is closed in Z and*

$$dim(Ker(D_x F(x_0, \lambda_0))) = codim(Range(D_x F(x_0, \lambda_0))) = 1$$

- *We have the "non-resonance" condition*

$$D_\lambda F(x_0, \lambda_0) \notin Range(D_x F(x_0, \lambda_0)) \tag{61}$$

Then there exists a continuously differentiable curve through (x_0, λ_0)

$$\{(x(r), \lambda(r)) | r \in (-\delta, \delta), (x(0), \lambda(0)) = (x_0, \lambda_0)\}$$

such that

$$F(x(r), \lambda(r)) = 0, \text{ for } r \in (-\delta, \delta)$$

for some $\delta > 0$ and all the solutions of $F(x, \lambda) = 0$ in a neighborhood of (x_0, λ_0) belong to the curve specified above.

The *"non-resonance"* condition (61) in our case is

$$\int_0^{2\pi} g(\theta)(\frac{1}{g(\theta)} - \frac{1}{2\pi} \int_0^{2\pi} g^{-1}(\sigma) d\sigma) d\theta \neq 0$$

or

$$(2\pi)^2 \neq \int_0^{2\pi} g(\sigma) d\sigma \cdot \int_0^{2\pi} g^{-1}(\sigma) d\sigma$$

Recalling that $g(\theta) = \frac{e^{r(b) \cos(2\theta)}}{Z(b)}$ with $Z(b) = \int_0^{2\pi} e^{r(b) \cos(2\theta)} d\theta$ the last expression becomes

$$(2\pi)^2 \neq (Z(b))^2$$

Let us observe that this is always true for $b > 4$. Indeed, let $h(x) = \int_0^{2\pi} e^{x \cos(2\theta)} d\theta$. Then

$$\frac{h'(r(b))}{h(r(b))} = \frac{2r}{b}$$

(see [6]).

Observe that $Z(b) = h(r(b))$ and multiplying the last equality by $\frac{dr}{db}$ we get

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$$\frac{Z'(b)}{Z(b)} = \frac{d}{db}(r^2)\frac{1}{b}$$

and taking into account that $r(4) = 0$ (see [6]) we have

$$Z(b) = 2\pi e^{\int_4^b (r^2)'(s)\frac{ds}{s}}$$

On the other hand, integrating by parts we have

$$\int_4^b (r^2)'(s)\frac{ds}{s} = \frac{r^2(b)}{b} - \underbrace{\frac{r^2(4)}{4}}_{=0} + \int_4^b \frac{r^2(s)}{s^2}ds > 0$$

which implies that $Z(b) \neq 2\pi$.

We want now to check that the solution thus obtained is positive and genuinely time dependent (i.e. not a stationary state in the initial reference frame). If λ is small enough this periodic solution will be near (pointwise in time) 0. More precisely we have

$$\|z\|_{L^\infty[0,\frac{\pi}{\omega}]\times[0,2\pi]} \leq C\|z\|_{\mathcal{X}} \tag{62}$$

for C independent of u, which is just Morrey's imbedding inequality (see for instance [9]).

Thus, for λ small enough, z is close to zero in L^∞ in time and space, and since g is positive, $z + g$ will be positive as well. Notice that we have worked in the moving frame. Moving back to the non-rotating frame we have that the solution f of the equation (36) is still time periodic and, pointwise in time, near $g(\theta + \omega t)$. Using the fact that $g(\theta)$ is nonconstant, this implies that for small enough λ this is a genuine time dependent periodic solution. Indeed, as g is nonconstant there exists $\theta_1 \neq \theta_2 \in [0, \pi]$ such that $g(\theta_1) \neq g(\theta_2)$. Take $\epsilon = \frac{1}{3}|g(\theta_2) - g(\theta_1)|$. Then, for λ small enough, we have $|\tilde{f}(t,\theta) - g(\theta)| < \epsilon, \forall \theta \in [0, 2\pi], \forall t \in [0, \frac{\pi}{\omega}]$ which recalling the definition of $\tilde{f}$ implies

$$|f(t, \theta_1 + \omega t) - g(\theta_1)| < \epsilon, \forall t \in [0, \frac{\pi}{\omega}]$$

$$|f(t, \theta_2 + \omega t) - g(\theta_2)| < \epsilon, \forall t \in [0, \frac{\pi}{\omega}]$$

Assume without loss of generality that $\theta_2 > \theta_1, \omega > 0$. Taking $t = \frac{\theta_2 - \theta_1}{\omega} < \frac{\pi}{\omega}$ in the first relation above and $t = 0$ in the second one, we obtain a contradiction if we assume that f is time independent. $\square$

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